Math 10B with Professor Stankova
Worksheet, Discussion #28; Tuesday, 5/7/2019
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1 Review Topics

1.1 Counting and Probability

1.
- Permutations and Combinations
  - Binomial coefficients
  - Poker-type problems
  - MISSISSIPPI problems
- Principle of Inclusion-Exclusion (PIE)
  - Complementary Counting
- Pigeonhole principle
- 12-Fold way
  - (In)distinguishable balls and (in)distinguishable boxes
  - Sterling Numbers of the Second kind
  - Partition Numbers
- Probability, Expected Value, Variance, Covariance
  - Random variable picture
  - Probability Mass Functions (PMF)
  - Formulas for expected values
  - Conditional probability
  - Independence
  - Chebyshev's Inequality
- Bayes' Theorem
- Distributions
  - Uniform Distribution
  - Bernoulli Trials
  - Binomial distribution
  - Hyper-geometric distribution
  - Geometric distribution
  - Poisson distribution
  - Normal distribution
  - t distribution

  - χ^2 distribution
  - Expected value and variance of each
- Maximum Likelihood
- Hypothesis Testing
  - Z Testing
  - t Testing
  - χ^2 Testing
  - Independence Testing
  - Null/Alternative Hypotheses
  - Type 1/type 2 errors, significance level, power
- Estimators and Confidence Intervals
  - Estimators for the mean and standard deviation
    * Biased/unbiased estimators
  - 95% confidence intervals

1.2 Miscellaneous

- Geometric sequences
- Induction
- Sorting Algorithms
  - Bubble Sort
  - Quick Sort
- Stable-matching algorithm
- Linear Regression
  - Least Squares Error
  - Finding line of best fit
- Correlation
- Gamma Function

2 Counting

2. **TRUE** False There are as many bit strings of length n as there are subsets of a set of n elements.

Solution: Both are 2^n.

3. True **FALSE** Complementary counting is always easier than the method of exhaustion.

> **Solution:** It is not always easier.

4. True **FALSE** I have socks of 3 different colors. If I randomly pick 2 of them, then PHP tells me that at least 2 of them must be the same color.

> **Solution:** In this case pigeons = socks and holes = colors so PHP says $\lceil 2/3 \rceil = 1$ of them is of the same color.

5. **TRUE** False We have $\binom{n}{r} = \binom{n}{n-r}$.

6. How many ways are there to rearrange the letters of $ZVEZDA$?

> **Solution:** $\frac{6!}{2!} = 360$ where we divide by 2! because there are 2 Z's.

7. How many ways can I split up 20 distinct people into 4 identical groups of 5?

> **Solution:** First we assume that the groups are distinguishable. There are $\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5}$ ways to do this. Then, since the groups are actually indistinguishable and identical, we need to divide by 4! just like a MISSISSIPPI problem. So the answer is
>
> $$\frac{\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5}}{4!}.$$

8. How many bit strings of length 10 begin with a 1 or end with a 1?

> **Solution:** We use PIE to get $2^9 + 2^9 - 2^8 = 2^{10} - 2^8$.

9. What is the coefficient of $x^2y^3z^5$ in $(x/2 + 3y - 2z)^{10}$? What about the coefficient of $x^3y^3z^3$?

> **Solution:** $\binom{10}{2,3,5}(\frac{1}{2})^2(3)^3(-2)^5 = \binom{10}{2}\binom{8}{3}\binom{5}{5}(27 \cdot -8) = \frac{10!}{2!3!5!} \cdot 27 \cdot (-8)$.

10. How many ways are there fore 8 men and 5 women to stand in a line so that no two women stand next to each other?

> **Solution:** We place the men first and in the spaces in between and the ends one woman can go. So there are 9 spots for the 5 women and the answer is $\binom{9}{5}$.

11. How many ways can we select 5 elements from a set of 3 elements if order matters and repetition is allowed?

> **Solution:** 3^5.

12. How many license plates of 3 letters followed by 3 numbers do not contain three of the same letters nor three of the same digit?

> **Solution:** There are $26^3 - 26$ ways to choose the 3 letters and $10^3 - 10$ ways to choose the digits. So $(26^3 - 26)(10^3 - 10)$.

13. How many ways can I choose 8 donuts from a box containing 20 identical glazed donuts and 30 identical chocolate ones?

> **Solution:** The 20 and 30 are red herrings and don't matter, all that matters is that there are 2 kinds. We are putting 8 indistinguishable balls into 2 distinguishable boxes so $\binom{9}{8} = 9$.

14. Prove that $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$ in two different ways.

> **Solution:** We can first prove it algebraically by
>
> $$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\ &= \frac{1}{n-r}\frac{(n-1)!}{(r-1)!(n-r-1)!} + \frac{1}{r}\frac{(n-1)!}{(r-1)!(n-r-1)!} \\ &= \frac{r+n-r}{r(n-r)}\frac{(n-1)!}{(r-1)!(n-r-1)!} \\ &= \frac{n!}{r!(n-r)!} = \binom{n}{r}.\end{aligned}$$

Then, to prove it combinatorially, we see that the right side is just choosing a team of r people out of n total people. We can do this by first distinguishing one person out, say Zvezda. Then, if she is on the team, out of the remaining $n-1$ people, we still need to choose $r-1$ which can be done in $\binom{n-1}{r-1}$ ways. Otherwise, if she is not on the team, we still need to choose r people which can be done in $\binom{n-1}{r}$ ways. Since both count the number of ways, the left and right side are equal.

15. How many ways can I create a license plate that has 3 letters followed by 3 numbers if I want exactly 1 I and at least 1 1.

Solution: First we count the number of ways we can choose 3 letters with exactly 1 I. The I can be in the first, second, or third spot. Each of the other options can be anything other than I and hence there are $1 \cdot 25 \cdot 25 + 25 \cdot 1 \cdot 25 + 25 \cdot 25 \cdot 1 = 3 \cdot 25^2$ different ways to choose the 3 letters.

Now we count the number of ways to pick the numbers. We see at least so we think complementary counting. There are 9^3 ways to choose 0 1's so there are $10^3 - 9^3$ ways to choose at least 1 1. Therefore, there are $(3 \cdot 25^2)(10^3 - 9^3)$ different license plates.

16. How many ways can I pack my 10 groceries into 3 identical bags if the bags must be non-empty?

Solution: $S(10, 3)$, the Stirling number of the second kind.

17. There are 2504 students at a school. Of them 1876 have taken Algebra, 999 have taken Biology, and 345 have taken chemistry. Moreover, 876 have taken both algebra and bio, 231 have taken bio and chem, and 290 have taken algebra and chem. Finally, 189 of them have taken all three. How many students have taken none of the three?

Solution: Using PIE, the number of students who have taken at least one of the classes is

$$1876 + 999 + 345 - 876 - 231 - 290 + 189 = 2012.$$

So $2504 - 2012 = 492$ have taken none of them.

18. How many 5 card hands out of a standard 52 card deck have 4 different suits?

Solution: If we want all 4 suits, 3 of the suits will have 1 card and one will have two cards in that suit. There are $\binom{4}{1} = 4$ ways to choose the suit that has 2 cards. Then for that suit, there are $\binom{13}{2}$ ways to choose the two cards in that suit. For the three remaining suit, we just need to pick one card in that suit and there are $\binom{13}{1} = 13$ ways to do this. So, the number of hands is

$$\binom{4}{1}\binom{13}{2}\binom{13}{1}^3.$$

19. I am baking cookies for Alice Bob and Carol. Each want at least 1 cookie but Bob wants at least 3 cookies. Alice is on a diet and wants at most 3 cookies. How many ways can I divide the 10 cookies I made amongst them?

Solution: Let x_1, x_2, x_3 be the number of cookies they get. Then our conditions are that $1 \le x_1 \le 3, 3 \le x_2, 1 \le x_3$. We can think of this as giving Alice and Carol 1 cookie and Bob 3 cookies so we have 5 cookies left to distribute and now Alice can only get two more cookies. So we have the problem $y_1 + y_2 + y_3 = 5$ with $0 \le y_1 \le 2$. We can do this by letting $y_1 = 0, 1, 2$ and calculating the number of ways for each and summing them together. Another way to do this is counting the total number of ways without the restriction $y_1 \le 2$ to get $\binom{5+3-1}{5} = \binom{7}{5}$ ways, and then subtracting the number of bad ways, which is when $y_1 \ge 3$. If $y_1 \ge 3$, we can subtract 3 to get $y_1' + y_2 + y_3 = 2$ and there are $\binom{2+3-1}{2} = \binom{4}{2}$ ways for this to happen. This gives a total of $\binom{7}{5} - \binom{4}{2} = 21 - 6 = 15$ total ways.

20. In the previous cookie problem, let X be the most number of cookies any one of Alice, Bob, or Carol gets (if they got $2, 3, 5$ cookies, then $X = 5$), what can we say about the minimum X can be?

Solution: By Pigeonhole Principle, by giving out 10 cookies to 3 people, there exists someone who gets at least $\lceil 10/3 \rceil = 4$ cookies. So, X must be at least 4.

3 Probability

21. **TRUE** False If there are only n possible outcomes in Ω, then there are 2^n inputs to the probability function P.

Solution: Any subset can be an input so there are 2^n subsets of Ω.

22. True **FALSE** For any two events we have $P(A|B) = P(B|A)$.
23. True **FALSE** For any two events A, B, we have $P(A \cup B) = P(A) + P(B)$.

Solution: PIE tells us that it should be $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

24. True **FALSE** If A, B are events with $P(A), P(B) > 0$, then $P(A|B) > P(A)$.

Solution: A, B could be disjoint so $P(A|B) = 0$.

25. **TRUE** False If A, B are independent events, then $P(A|B) = P(A)$.
26. **TRUE** False If A, B are independent events, then $\bar{A}$ and B are independent.
27. True **FALSE** Two random variables X, Y are independent if for all a, b we have $P(XY = ab) = P(X = a) \cdot P(Y = b)$.

Solution: We need to show for all a, b that $P(X = a \cap Y = b) = P(X = a)P(Y = b)$.

28. Let X be a random variable on a probability space Ω with a probability function P and let f be the PMF for X. Draw a picture of how all these variables interact and explain any special arrows that you have in your diagram. Do the same for if X is a continuous random variable.

Solution: P is a dashed line from Ω to $[0, 1]$ because it takes in subsets of Ω. X is a solid arrow from Ω to $\mathbb{R}$ because it is a function that takes outcomes to a value. f is a solid arrow from $\mathbb{R}$ to $[0, 1]$.

29. In Berkeley, it is either sunny or cloudy. Suppose that it is cloudy with probability 30% and when it is cloudy, it rains with probability 80%. The probability that it rains with it is sunny is 1%. What is the probability that it rains in Berkeley?

Solution:

$$P(rain) = P(rain|sunny)P(sunny) + P(rain|cloudy)P(cloudy)$$

$$= 0.01 \cdot 0.7 + 0.8 \cdot 0.3 = 0.007 + 0.24 = 0.247.$$

30. Suppose that our outcome space is $\{1, 2, 3, 4\}$ with $P(\{1\}) = P(\{3\}) = 16\%, P(\{2\}) = 4\%, P(\{4\}) = 64\%$. Are $\{1, 2\}$ and $\{2, 3\}$ independent?

Solution:

$$P(\{1, 2\} \cap \{2, 3\}) = P(\{2\}) = 4\%.$$

$$P(\{1, 2\})P(\{2, 3\}) = (16\% + 4\%)(64\% + 4\%) = 0.2 \cdot 0.68 = 0.136 \neq 4\%.$$

So no.

31. Suppose you play a lottery where 6 numbers are selected out of the numbers 1 to 40 inclusive. You pick 6 numbers, what is the probability that only one of them is correct?

Solution: First we choose the correct number which is $\binom{6}{1}$ ways then choose 5 wrong numbers out of the $40 - 6 = 34$ total to get $\binom{34}{5}$. Finally, we divide by the total number of ways to choose the ticket $\binom{40}{6}$ to get

$$\frac{\binom{6}{1}\binom{34}{5}}{\binom{40}{6}}.$$

32. Eve has 5 cards in her hand and I know that one of them is the ace of spades. What is the probability that she has a pair of aces (exactly 2 aces)?

Solution: Let A be the event that she has a pair of aces and let B be the event that she has the ace of spades. Then we want to calculate $P(A|B)$. By definition, this is $\frac{P(A\cap B)}{P(B)}$. The probability $P(B) = \frac{\binom{51}{4}}{\binom{52}{5}}$ because we know we have the ace of spades and then we just need to pick 4 other cards from the remaining 51 cards to fill out our hand. Then $P(A \cap B)$ is the probability of having a pair of aces with one of them being the ace of spades. To count this, we know that we have the ace of spades so there are $\binom{3}{1}$ ways to choose the other ace to complete the pair. Then out of the remaining 48 non-ace cards, we need to pick out 3 cards to fill out our hand. Therefore $P(A \cap B) = \frac{\binom{3}{1}\binom{48}{3}}{\binom{52}{5}}$. Therefore

$$P(A|B) = \frac{\binom{3}{1}\binom{48}{3}}{\binom{51}{4}}.$$

33. You ask two friends for a favor. The first friend has a 10% chance of saying yes and the second has a 5% chance. The probability that they both say yes is 3%. What is the probability that at least one friend will help you?

Solution: Use PIE to get $10\% + 5\% - 3\% = 12\%$.

34. A red-green colorblind person picks an apple out of a bag. There are 4 red apples and 1 green apple. With probability 3/4 he says the correct color of the apple he picked out. What is the probability that he says that the apple he picks out is red?

Solution: The probability that he says red is the probability that the apple is red and he says red plus the probability that the apple is green and he says red. The first probability is $\frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$. The second probability is $\frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$. So the probability that he says red is $\frac{3}{5} + \frac{1}{20} = \frac{13}{20}$.

35. In the previous apple problem, what is the probability that the apple is actually red when he says it is red?

Solution: Let A be the event that the apple is red and let B be the event that he says red. Then we want to calculate $P(A|B) = \frac{P(A \cap B)}{P(B)}$. We calculated $P(B) = \frac{13}{20}$ from the previous problem. Then $P(A) = \frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$. So $P(A|B) = \frac{12}{13}$. You can also solve this using Bayes Theorem.

36. You have a bag containing 3 fair 6 sided die and one biased one so that the probability of rolling 1 is 0.5 and rolling $2-6$ each is 10%. You randomly select a die and roll the die to get a 1. What is the probability you selected a fair die?

Solution: Using Bayes

$$\begin{aligned} P(fair|1) &= \frac{P(1|fair)P(fair)}{P(1|fair)P(fair) + P(1|unfair)P(unfair)} \\ &= \frac{1/6 \cdot 3/4}{1/6 \cdot 3/4 + 1/2 \cdot 1/4} \\ &= \frac{1/8}{1/8 + 1/8} = \frac{1}{2}. \end{aligned}$$

4 Discrete Distributions

37. True **FALSE** Doing n independent Bernoulli trials and counting the number of successes is a geometric distribution.

Solution: It is binomial.

38. **TRUE** False If X is a Poisson variable with parameter 1 and Y is Poisson with parameter 2, then the probability that $Y = 2$ is larger than the probability $X = 2$.

Solution: $P(Y = 2) = \frac{2^2 e^{-2}}{2!} = 2e^{-2}$. $P(X = 2) = \frac{1^2 e^{-1}}{2!} = e^{-2}/2$. But $4 > e$ so $P(Y = 2) > P(X = 2)$.

39. True **FALSE** If $\mu = E[X]$, then $P(X = \mu)$ is higher than $P(X = k)$ for all other k.

Solution: $P(X = \mu)$ could be 0 (think die roll).

40. True **FALSE** If X, Y are random variables with $Cov(X, Y) = 0$, then X, Y are independent.

41. **TRUE** False A constant random variable has variance 0.

42. **TRUE** False If X has standard error σ and expected value μ, then $10X$ has expected value 10μ and standard error 10σ.

43. True **FALSE** It is possible to have a Poisson random variable with expected value 1 and variance 2.

Solution: For a Poisson distribution we have expected value equal to variance.

44. Suppose that I have a weighted die that lands on $1, 2, 3, 4, 5$ with equal probability and 6 5 times as likely as 1. Let X be the value of the die. What is the PMF for X?

Solution: $f(1) = f(2) = f(3) = f(4) = f(5) = \frac{1}{10}, f(6) = \frac{1}{2}$.

45. A detective is gathering information about a bank robbery by interviewing citizens. Suppose that out of the 300 citizens in the town, 20 of them witnessed the crime. What is the probability that the detective interviews exactly 3 witnesses if she interviews 50 random distinct people?

Solution: This is hypergeometric with $N = 300, n = 50, m = 20, k = 3$ to get

$$\frac{\binom{20}{3}\binom{280}{47}}{\binom{300}{50}}.$$

46. For my weighted die in the previous problem, what is the probability in 10 rolls, I roll a 5 or 6 exactly 6 times? What kind of distribution is this?

Solution: The probability of rolling a 5 or 6 is $\frac{1}{10} + \frac{1}{2} = \frac{3}{5}$. This is a binomial distribution and the probability is $\binom{10}{6}(3/5)^6(2/5)^4$.

47. For my weighted die in the previous problem, suppose that I keep rolling until I roll a 5 or 6. What is the expected number of times I need to roll the die? What kind of distribution is this?

Solution: This is a geometric distribution. The expected number of times I need to roll the die, including the last roll, is $1 + \frac{1-p}{p} = 1 + \frac{2/5}{3/5} = \frac{5}{3}$.

48. Suppose that X is binomially distributed with $E[X] = 15$ and $Var(X) = 6$. How many trials n are there and what is the probability p of success?

Solution: Since this is binomial, we know $np = 15$ and $np(1-p) = 6$ so $1-p = 2/5$ and $p = 3/5$ so $n = 15/(3/5) = 25$.

49. Suppose the number of frogs seen at a pond each day is Poisson distributed with an average of 0.1 per day. What is the probability you see more than 1 frog?

Solution: $P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{\lambda^0 e^{-\lambda}}{0!} - \frac{\lambda^1 e^{-\lambda}}{1!} = 1 - (\lambda + 1)e^{-\lambda} = 1 - 1.1e^{-0.1}$.

50. Suppose that the number of students who fill out course evaluations per day is Poisson distributed and on average 2 students fill out evaluations per day. What is the probability that in a week, no students fill out evaluations? What is the probability that in a week, 7 people fill out evaluations?

Solution: In a week, the number of students that fill out evaluations is Poisson distribution and on average there are $2 \cdot 7 = 14$ students that fill out evals. So $\lambda = 14$. The probability of having 0 students fill out evals is $f(0) = \frac{14^0 e^{-14}}{0!} = e^{-14}$. The probability of 7 students filling out evals is $f(7) = \frac{14^7 e^{-14}}{7!}$.

51. When I roll a fair 6 sided die 10 times, what is the expected number of distinct numbers that appear? (For instance, if I roll $1, 1, 3, 3, 2$, there are 3 distinct numbers that appear)

Solution: Let $X_1, X_2, \ldots, X_6$ be random variables such that $X_i = 1$ if I roll an i in the 10 times and 0 otherwise. Then the number of distinct numbers that appear is $E[X_1 + X_2 + \cdots + X_6] = E[X_1] + E[X_2] + \cdots + E[X_6]$. Then $E[X_1]$ is just the probability that I roll at least 1 1 in 10 roll. We calculate this via complementary counting because it says at least. This probability is $1 - P(A)$ where A is the event that I roll 0 1s. The probability of this is $P(A) = (\frac{5}{6})^{10}$. So the expected number of distinct numbers is $6(1 - (5/6)^{10})$.

52. Let $X_1, \ldots, X_4$ be i.i.d Bernoulli trials with $p = \frac{3}{4}$. Let $\bar{X}$ be the average of them. What is $Var[\bar{X}]$? Find $Cov(X_1, \bar{X})$ (Hint: Write $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$).

Solution: Each of the X_i is Bernoulli so expected value of $p = \frac{3}{4}$ and variance of $p(1-p) = \frac{3}{16}$. Then the variance of $Var[\bar{X}] = Var[X_1]/n = \frac{3}{16 \cdot 4} = \frac{3}{64}$. Finally, we have that

$$\begin{aligned} Cov(X_1, \bar{X}) &= Cov(X_1, \frac{1}{4}(X_1 + X_2 + X_3 + X_4)) \\ &= \frac{1}{4}(Cov(X_1, X_1) + Cov(X_1, X_2) + Cov(X_1, X_3) + Cov(X_1, X_4)) \\ &= \frac{1}{4}(Var(X_1) + 0 + 0 + 0) \\ &= \frac{1}{4}\frac{3}{16} = \frac{3}{64}. \end{aligned}$$

5 Miscellaneous

53. Use MMI to prove that $1 + \frac{1}{4} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n > 1$.

Solution: So the base case of $n = 2$ then

$$\begin{aligned} 1 + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} &< 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ &= 2 - (\frac{1}{n} - \frac{1}{(n+1)^2}) \\ &< 2 - \frac{1}{n+1}. \end{aligned}$$

54. Let $a_{n+1} = a_n + 2a_{n-1}$ with $a_0 = 2$ and $a_1 = 1$. Use MMI to prove that $a_n = 2^n + (-1)^n$ for all $n \geq 0$.

Solution: There are two base cases of $a_0 = 2^0 + (-1)^0 = 1 + 1 = 2$ and $a_1 = 2^1 + (-1)^1 = 2 - 1 = 1$. Now assume that $a_n = 2^n + (-1)^n$ **and** $a_{n+1} = 2^{n+1} + (-1)^{n+1}$

for some $n \geq 0$. We want to now show S_{n+2}, that $a_{n+2} = 2^{n+2} + (-1)^{n+2}$. It goes as

$$\begin{aligned} a_{n+2} &= a_{n+1} + 2a_n \\ &= 2^{n+1} + (-1)^{n+1} + 2 \cdot 2^n + 2 \cdot (-1)^n \\ &= 2^{n+1} + 2^{n+1} + 2 \cdot (-1)^n + (-1)(-1)^n \\ &= 2^{n+1+1} + (-1)^n \\ &= 2^{n+2} + (-1)^2(-1)^n \\ &= 2^{n+2} + (-1)^{n+2} \end{aligned}$$

So by MMI we have proved that $a_n = 2^n + (-1)^n$ for all $n \geq 0$.

55. Suppose that three people randomly pick a hat. What is the expected value of the number of people who choose their hat? (with proof). What is the variance?

Solution: Use $X = X_1 + X_2 + X_3$ where X_i is Bernoulli on whether person i gets their hat back. Then we calculate $E[X_i] = \frac{1}{3}$ and $E[X_i X_j] = \frac{1}{6}$ for $i \neq j$. This gives that expected value and variance are both 1.